Research Article

Some Orlicz Space of Entire Sequence Spaces

Naveen Kumar Srivastava
St. Andrew's College, Gorakhpur, Uttar Pradesh, India

Corresponding Author: *Naveen Kumar Srivastava

DOI: https://doi.org/10.5281/zenodo.19508681

Abstract

Let Γ denote the space of all entire sequences and $\wedge$ the space of all analytic sequences. This paper is devoted to the study of the general properties of the Orlicz space of.

Manuscript Information

- **ISSN No:** 2583-7397
- **Received:** 13-02-2026
- **Accepted:** 28-03-2026
- **Published:** 11-04-2026
- **IJCRM:**5(2); 2026: 499-509
- **©2026, All Rights Reserved**
- **Plagiarism Checked:** Yes
- **Peer Review Process:** Yes

How to Cite this Article

Srivastava N K. Some Orlicz Space of Entire Sequence Spaces. Int J Contemp Res Multidiscip. 2026;5(2):499-509.

Access this Article Online

www.multiarticlesjournal.com

KEYWORDS: Sequence Spaces, Summability Methods, Functional Analysis, Orlicz Spaces, Modulus Function

1. INTRODUCTION

An Orlicz function is a function $M : [0, \infty) \rightarrow [0, \infty)$ which is continuous**,** non-decreasing, and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$, and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$. If the convexity of the Orlicz function M is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called a modulus function, defined and discussed by Ruckle [5] and Maddox [4].

An Orlicz function M is said to satisfy the Δ_2-condition for all values of u if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)$ $(u \geq 0)$. The Δ_2-condition is equivalent to $M(\ell u) < K. \ell M(u)$, for all values of u and for $\ell > 1$. (λ_k) and (μ_k) is a sequence of nonzero complex numbers.

An Orlicz function M can always be represented in the following integral form: $M(x) = \int_0^x q(t)\,dt$, where q, known as the kernel of M, is right-differentiable for $t \geq 0$, $q(0) = 0$, $q(t) > 0$ for $t > 0$, q is non-decreasing, and $q(t) \to \infty$ as $t \to \infty$. Lindenstrauss and Tzafriri [3] used the idea of an Orlicz function to construct the Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{\lambda_k x_k}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}, \qquad \dots (1.1)$$

where w = {all complex sequences}.

The space ℓ_M with the norm

$$\| x \| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{\lambda_k x_k}{\rho}\right) \leq 1 \right\} \qquad \dots (1.2)$$

becomes a Banach space which is called an Orlicz sequence space.

2. Complex sequence

A complex sequence whose k^{th} term is x_k will be denoted by (x_k) or x. A sequence $x = (x_k)$ is said to be analytic if $|\lambda_k x_k|^{1/k} < \infty$. The vector space of all analytic sequences will be denoted by Λ. A sequence x is called an entire sequence if $\lim_{k\to\infty} |\lambda_k x_k|^{1/k} = 0$. The vector space of all entire sequences will be denoted by Γ.

Definition 2.1 : The space consisting of all sequences x in w such that $M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right) \to 0$ ask $\to \infty$ for some arbitrarily fixed $\rho > 0$ is denoted by Γ_M, with M being a modulus function. In other words, $\left\{ M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right)\right\}$ is a null sequence. The space Γ_M is a metric space with the metric

$$d(x, y) = \sup_{\{k\}} M\left(\frac{|\lambda_k x_k - \mu_k y_k|^{1/k}}{\rho}\right) \qquad \dots (2.1)$$

for all $x = \{x_k\}$ and $y = (y_k)$ in Γ_M.

Given a sequence $x = (x_k)$ whose n^{th} section is the sequence $x^{(n)} = \{x_1, x_2, \dots, x_n, 0, 0, \dots\}$, $\delta^{(n)} = (0, 0, \dots, 1, 0, 0, \dots)$, with 1 in the n^{th} place and zeros elsewhere; let Φ = {all finite sequences}.

An FK-space (or a metric space) X is said to have AK property if ($\delta^{(n)}$) is a Schauder basis for X. Or equivalently $x^{(n)} \to x$.

The space is said to have or be an AD space if Φ is dense in X.

We note that AK implies AD by [1].

If X is a sequence space, we give the following definitions:

(i) X' the continuous dual of X;

(ii) $X^{\alpha} = \{a = (a_k)$: $\sum_{k=1}^{\infty} |a_k \lambda_k x_k| < \infty$, for each $x \in X\}$;

(iii) $X^{\beta} = \{a = (a_k)$: $\sum_{k=1}^{\infty} a_k \lambda_k x_k$ is convergent, for each $x \in X\}$;

(iv) $X^{\gamma} = \{a = (a_k) : \sup_{(n)} \left|\sum_{k=1}^{n} a_k \lambda_k x_k\right| < \infty$, for each $x \in X\}$;

(v) let X be an FK-space $\supset \Phi$, then $X^f = \{f(\delta^{(n)}) : f \in X')$. X^{α}, X^{β}, and X^{γ} are called the α – (or Köthe-Toeplitz) dual of X, β – (or generalized-Köthe-Toeplitz) dual of X, and γ -dual of X, respectively.

Note that $X^{\alpha} \subset X^{\beta} \subset X^{\gamma}$. If $X \subset Y$, then $Y^{\mu} \subset X^{\mu}$, for $\mu = \alpha$, β, or γ.

Lemma 2.2: (see [6, Theorem 7.2.7]). Let X be an FK-space $\supset \Phi$. Then

(i) $X^{\gamma} \subset X^f$;

(ii) if X has AK, $X^{\beta} = X^f$;

(iii) if X has AD, $X^{\beta} = X^{\gamma}$.

We note that $\Gamma^{\alpha} = \Gamma^{\beta} = \Gamma^{\gamma} = \wedge$.

Proposition 2.3 : $\Gamma \subset \Gamma_M$, with the hypothesis that $M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right) \leq |\lambda_k x_k|^{1/k}$.

Proof: Let $X \in \Gamma$. Then we have the following implications:

$$|\lambda_k x_k|^{1/k} \to 0 \text{ as } k \to \infty. \qquad \dots (2.2)$$

But $M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right) \leq |\lambda_k x_k|^{1/k}$, by our assumption, implies that

$$M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right) \to 0 \text{ as } k \to \infty \text{ (by (2.2))}$$

$$\Rightarrow x \in \Gamma_M \qquad \dots (2.3)$$

$$\Rightarrow x \subset \Gamma_M.$$

This completes the proof.

Proposition 2.4: Γ_M has AK, where M is **a** modulus function.

Proof : Let $x = \{x_k\} \in \Gamma_M$, but then $\left\{M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right)\right\} \in \Gamma$, and hence

$$\sup_{k \geq n+1} M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right) \to 0 \text{ as } n \to \infty \qquad \dots (2.4)$$

By using (2.4), $d(x, x^{[n]}) = \sup_{k \geq n+1} M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right) \rightarrow 0$ as $n \rightarrow \infty$, which implies that $x^{[n]} \rightarrow x$ as $n \rightarrow \infty$, implying that Γ_M has AK. This completes the proof.

Proposition 2.5: Γ_M is solid.

Proof: Let $|x_k| \leq |y_k|$ and let $y = (y_k) \in \Gamma_M$. $M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right) \leq M\left(\frac{|\lambda_k y_k|^{1/k}}{\rho}\right)$, because M is non-decreasing. But $M\left(\frac{|\lambda_k y_k|^{1/k}}{\rho}\right) \in \Gamma$ because $y \in \Gamma_M$. That is, $M\left(\frac{|\lambda_k y_k|^{1/k}}{\rho}\right) \rightarrow 0$ as $k \rightarrow \infty$ and $M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right) \rightarrow 0$ as $k \rightarrow \infty$. Therefore $x = \{x_k\} \in \Gamma_M$. This completes the proof.

Proposition 2.6: Let M be an Orlicz function which satisfies the Δ_2-condition. Then $\Gamma \subset \Gamma_M$.

Proof: Let

$$x \in \Gamma \qquad \dots (2.5)$$

Then $|\lambda_k x_k|^{1/k} \leq \varepsilon$ for sufficiently large k and every $\varepsilon > 0$. But then by taking $\rho \geq 1/2$,

$$M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right) \leq M\left(\frac{\varepsilon}{\rho}\right) \text{ (because M is non-decreasing)}$$

$$\leq M(2\varepsilon)$$

$$\Rightarrow M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right) \leq KM(\varepsilon) \text{ (by the A2-condition, for some } K > 0) \qquad \dots (2.6)$$

$$\leq \varepsilon \quad \text{(by defining } M(\varepsilon) < \frac{\varepsilon}{k})$$

$$\Rightarrow M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Hence, $x \in \Gamma_M$.

From (2.5) and since

$$x \in \Gamma_M, \qquad \dots (2.7)$$

we get

$$\Gamma \subset \Gamma_M, \qquad \dots (2.8)$$

This completes the proof.

Proposition 2.7: If M is a modulus function, then Γ_M is a linear set over the set of complex numbers C.

Proof : Let $x, y \in \Gamma_M$ and $\alpha, \beta \in$ C. In order to prove the result, we need to find some ρ_3 such that

$$M\left(\frac{|\alpha\lambda_k x_k+\beta\mu_k y_k|^{1/k}}{\rho_3}\right)\to 0 \quad \text{as } k\to\infty \qquad \ldots (2.9)$$

Since x, y $\in \Gamma_M$, there exist some positive ρ_1 and ρ_2 such that

$$M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho_1}\right)\to 0 \quad \text{as } k\to\infty, \qquad \ldots (2.10)$$

$$M\left(\frac{|\lambda_k y_k|^{1/k}}{\rho_2}\right)\to 0 \quad \text{as } k\to\infty.$$

Since M is a non-decreasing modulus function, we have

$$M\left(\frac{|\alpha\lambda_k x_k+\beta\mu_k y_k|^{1/k}}{\rho_3}\right) \leq M\left(\frac{|\alpha\lambda_k x_k|^{1/k}}{\rho_3}+\frac{|\beta\mu_k y_k|^{1/k}}{\rho_3}\right)$$

$$\leq M\left(\frac{|\alpha|^{1/k}|\lambda_k x_k|^{1/k}}{\rho_3}+\frac{|\beta|^{1/k}|\mu_k y_k|^{1/k}}{\rho_3}\right) \qquad \ldots (2.11)$$

$$\leq M\left(\frac{|\alpha|\ \ |\lambda_k x_k|^{1/k}}{\rho_3}+\frac{|\beta_k|\ \ |\mu_k y_k|^{1/k}}{\rho_3}\right).$$

Take ρ_3 such that

$$\frac{1}{\rho_3} = \min\left\{\frac{1}{|\alpha|}\frac{1}{\rho_1}, \frac{1}{|\beta|}\frac{1}{\rho_2}\right\} \qquad \ldots (2.12)$$

Then

$$M\left(\frac{|\alpha\lambda_k x_k+\beta\mu_k y_k|^{1/k}}{\rho_3}\right) \leq M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho_1}+\frac{|\mu_k y_k|^{1/k}}{\rho_2}\right)$$

$$\leq M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho_1}\right)+M\left(\frac{|\mu_k y_k|^{1/k}}{\rho_2}\right) \qquad \ldots (2.13)$$

$\to$ 0 (by (2.10))

Hence,

$$M\left(\frac{|\alpha\lambda_k x_k+\beta\mu_k y_k|^{1/k}}{\rho_3}\right)\to 0 \quad \text{as } k\to\infty \qquad \ldots (2.14)$$

So (αx+ β y) $\in \Gamma_M$. Therefore, Γ_M is linear. This completes the proof.

Definition 2.8: Let p = (p_k) be any sequence of positive real numbers. Then

$$\Gamma_M(p) = \left\{ x = \{x_k\} : \left(M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho} \right) \right)^{p_k} \to 0 \text{ as } k \to \infty \right\} \qquad \ldots (2.15)$$

Suppose that p_k is a constant for all k, then $\Gamma_M(p) = \Gamma_M$.

Proposition 2.9: Let $0 \le p \le q_k$ and let $\{q_k/p_k\}$ be bounded. Then $\Gamma_M(q) \subset \Gamma_M(p)$.

Proof: Let

$$x \in \Gamma_M(q) \qquad \ldots (2.16)$$

$$\left(M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho} \right) \right)^{q_k} \to 0 \quad \text{as} k \to \infty \qquad \ldots (2.17)$$

Let $t_k = \left(M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho} \right) \right)^{q_k}$ and $\gamma_k = \frac{p_k}{q_k}$. Since $p_k \le q_k$, we have $0 \le \gamma_k \le 1$.

Take $\mathbf{0} < \lambda < \lambda_k$. Define

$$u_k = \begin{cases} t_k & (t_k \ge 1) \\ 0 & (t_k < 1) \end{cases}$$

$$v_k = \begin{cases} 0 & (t_k \ge 1) \\ t_k & (t_k < 1) \end{cases} \qquad \ldots (2.18)$$

$$t_k = u_k + v_k, \qquad t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}.$$

Now it follows that

$$u_k^{\lambda_k} = u_k \le t_k, \qquad v_k^{\lambda_k} \le v_k^{\lambda} \qquad \ldots (2.19)$$

Since $t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$, then $t_k^{\lambda_k} \le t_k + v_k^{\lambda}$.

$$\left(\left(M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho} \right) \right)^{q_k} \right)^{\gamma_k} \le \left(\left(M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho} \right) \right)^{q_k} \right)^{\gamma_k'}$$

$$\Rightarrow \left(\left(M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho} \right) \right)^{q_k} \right)^{p_k/q_k} \le \left(M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho} \right) \right)^{q_k} \qquad \ldots (2.20)$$

$$\Rightarrow \left(M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho} \right) \right)^{p_k} \le \left(M\left(\frac{|\mu_k x_k|^{1/k}}{\rho} \right) \right)^{q_k}.$$

But

$$\left(M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right)\right)^{q_k} \to 0 \quad \text{(by (2.17))} \qquad \dots (2.21)$$

Hence $\left(M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right)\right)^{p_k} \to 0$ as $k \to \infty$. Hence

$$x \in \Gamma_M(p) \qquad \dots (2.22)$$

From (2.16) and (2.22), we get

$$\Gamma_M(q) \subset \Gamma_M(p) \qquad \dots (2.23)$$

This completes the proof.

Proposition 2.10: (a) Let $0 < \inf p_k \le p_k \le 1$. Then $\Gamma_M(p) \subset \Gamma_M$.

(b) Let $1 \le P_k \le \sup p_k < \infty$. Then $\Gamma_M \subset \Gamma_M(p)$.

Proof: (a) Let $x \in \Gamma_M(p)$,

$$\left(M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right)\right)^{p_k} \to 0 \text{ as } k \to \infty \qquad \dots (2.24)$$

Since $0 < \inf p_k \le p_k \le 1$,

$$\left(M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right)\right) \le \left(M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right)\right)^{p_k} \qquad \dots (2.25)$$

From (2.24) and (2.25) it follows that

$$x \in \Gamma_M \qquad \dots (2.26)$$

Thus,

$$\Gamma_M(p) \subset \Gamma_M \qquad \dots (2.27)$$

We have thus proven (a).

(b) Let $p_k \ge 1$ for each k and $\sup p_k < \infty$ and let $x \in \Gamma_M$.

$$M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right) \to 0 \text{ as } k \to \infty \qquad \dots (2.28)$$

Since $1 \le p_k \le \sup p_k < \infty$, we have

$$\left(M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right)\right)^{p_k} \le \left(M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right)\right) \qquad \ldots (2.29)$$

$$\left(M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right)\right)^{p_k} \to 0 \text{ as } k \to \infty \text{ (by using (2.28)).}$$

Therefore $x \in \Gamma_M(p)$. This completes the proof.

Proposition 2.11: Let $0 < p_k \le q_k < \infty$ for each k. Then $\Gamma_M(p) \subseteq \Gamma_M(q)$.

Proof: Let $x \in \Gamma_M(p)$

$$\left(M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right)\right)^{p_k} \to 0 \text{ as } k \to \infty \qquad \ldots (2.30)$$

This implies that $\left(M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right)\right) \le 1$ for sufficiently large k. Since M is non-decreasing, we get

$$\left(M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right)\right)^{q_k} \le \left(M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right)\right)^{p_k} \qquad \ldots (2.31)$$

$$\Rightarrow \left(M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right)\right)^{q_k} \to 0 \text{ as } k \to \infty \text{ (by using (2.30)).}$$

Since $x \in \Gamma_M(q)$, hence $\Gamma_M(p) \subseteq \Gamma_M(q)$. This completes the proof.

Proposition 2.12: $\Gamma_M(p)$ is r-convex for all r, where $0 \le r \le \inf p_k$. Moreover, if $p_k = p \le 1$ for all k, then they are p-convex.

Proof: We will prove the theorem for $\Gamma_M(p)$. Let $x \in \Gamma_M(p)$ and $r \in (0, \lim_{n\to\infty} \inf p_n)$. Then, there exists k_0 such that $r \le p_k$ for all $k > k_0$.
Now, define

$$g^*(x) = \inf\left\{\rho : M\left(\frac{|\lambda_k x_k - \mu_k y_k|^{1/k}}{\rho}\right)^r + M\left(\frac{|\lambda_k x_k - \mu_k y_k|^{1/k}}{\rho}\right)^{p_n}\right\} \qquad \ldots (2.32)$$

Since $r \le p \le 1$ for all $k > k_0$, g^* is sub-additive. Further, for $0 \le |\lambda| \le 1$,

$$|\gamma|^{p_k} \le |\gamma|^r \quad \forall\, k > k_0. \qquad \ldots (2.33)$$

Therefore, for each λ, we have

$$g^*(\gamma x) \le |\gamma|^r .g^*(x) \qquad \ldots (2.34)$$

Now, for $0 < \delta < 1$,

$$U = \{x : g^*(x) \le \delta\} \qquad \ldots (2.35)$$

which is an absolutely γ-convex set, for

$$|\gamma|^r + |\gamma|^r \le 1,\ x, y \in U \qquad \ldots (2.36)$$

Now,

$$
\begin{aligned}
g^*(\gamma x + \gamma' y) &\le g^*(\gamma x) + g^*(\gamma' y) \\
&\le |\gamma|^r g^*(x) + |\gamma'|^r g^*(y) \\
&\le |\gamma|^r \delta + |\gamma'|^r \delta \quad \text{(using (2.34) and 2.35)} \\
&\le |\gamma|^r + |\gamma'|^r \delta \\
&\le 1.\ \delta \quad \text{(by using (2.36))} \\
&\le \delta.
\end{aligned}
$$

If $p_k = p \le 1$ for all k, then for $0 < r < 1$, $U = (x: g^*(x) \le \delta)$ is an absolutely p-convex set. This can be obtained by a similar analysis, and therefore, we omit the details. This completes the proof.

Proposition 2.13: $(\Gamma_M)^\beta = \wedge$.

Proof:

Step 1 : $\Gamma \subset \Gamma_M$ by Proposition 2.3; this implies that $(\Gamma_M)^\beta \subset \Gamma^\beta = \wedge$. Therefore,

$$(\Gamma_M)^\beta = \wedge \qquad \ldots (2.38)$$

Step 2 : Let $y \in \wedge$. Then $|\mu_k y_k| < M^k$ for all k and for some constant $M > 0$.

Let $x \in \Gamma_M$. Then $M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right) \to 0$ ask $\to \infty$. Hence, $M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right) < \varepsilon$ for given $\varepsilon > 0$ for sufficiently large k.

Take $\varepsilon = \frac{1}{2}$ M so that $M\left(\frac{|\lambda_k x_k|}{\rho}\right) < \frac{1}{(2M)^k}$.

But then $M\left(\frac{|\lambda_k x_k \cdot \mu_k y_k|}{\rho}\right) \le \frac{1}{2^k}$ so that $\sum_{k=1}^{\infty} M\left(\frac{|\lambda_k x_k \cdot \mu_k y_k|}{\rho}\right)$ converges. Therefore, $\sum_{k=1}^{\infty} M\left(\frac{|\lambda_k x_k \cdot \mu_k y_k|}{\rho}\right)$ converges. Hence, $\sum_{k=1}^{\infty} \lambda_k x_k \cdot \mu_k y_k$ converges so that $y \in (\Gamma_M)^\beta$. Thus

$$\wedge \subset (\Gamma_M)^\beta \qquad \ldots (2.39)$$

Step 3: From (2.38) and (2.39), we obtain

$$(\Gamma_M)^{\beta} = \wedge \qquad \dots (2.40)$$

This completes the proof.

Proposition 2.14 : $(\Gamma_M)^{\mu} = \wedge$ for $\mu = \alpha, \beta, \gamma, f$.

Proof:

Step 1: Γ_M has AK by Proposition 2.4. Hence, by Lemma 2.2(i), we get $(\Gamma_M)^{\beta} = (\Gamma_M)^{f}$. But $(\Gamma_M)^{\beta} = \wedge$. Hence,

$$(\Gamma_M)^{f} = \wedge \qquad \dots (2.41)$$

Step 2 : Since AK implies AD, hence by Lemma 2.2(iii) we get $(\Gamma_M)^{\beta} = (\Gamma_M)^{\gamma}$. Therefore,

$$(\Gamma_M)^{\gamma} = \wedge \qquad \dots (2.42)$$

Step 3: Γ_M is normal by Proposition 2.5. Hence, by [2, Proposition 2.7], we get

$$(\Gamma_M)^{\alpha} = (\Gamma_M)^{\gamma} = \wedge \qquad \dots (2.43)$$

From (2.41), (2.42), and (2.43), we have

$$(\Gamma_M)^{\alpha} = (\Gamma_M)^{\beta} = (\Gamma_M)^{\gamma} = (\Gamma_M)^{f} = \wedge \qquad \dots (2.44)$$

Proposition 2.15: The dual space of Γ_M is $\wedge$. In other words, $\Gamma_M^{*} = \wedge$.

Proof: We recall that δ^k has 1 in the kth place and zeros elsewhere, with

$$x = \delta^k, \left\{M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right)\right\} = \left\{\frac{M(0)^1}{\rho}, \frac{M(0)^{1/2}}{\rho}, \dots, \frac{M(1)^{1/k}}{\rho}, \frac{M(0)^{1/(k+1)}}{\rho}, \dots\right\} = \left\{0, 0, \dots, \frac{M(1)^{1/k}}{\rho}, 0, \dots\right\} \qquad \dots (2.45)$$

which is a null sequence. Hence, $\delta^k \in \Gamma_M$. $f(x) = \sum_{k=1}^{\infty} \lambda_k x_k \cdot \mu_k y_k$ with $x \in \Gamma_M$ and $f \in \Gamma_M^{*}$, where Γ_M^{*} is the dual space of Γ_M. Take $x = \delta^k \in \Gamma_M$. Then

$$|\mu_k y_k| \leq \|f\|d(\delta^k, 0) < \infty \quad \forall k \qquad \dots (2.46)$$

Thus (y_k) is a bounded sequence and hence an analytic sequence. In other words, $y \in \wedge$. Therefore $\Gamma_M^{*} = \wedge$. This completes the proof.

Lemma 2.16: [6, Theorem 8.6.1]. $Y \supset X \Leftrightarrow Y^f \subset X^f$, where X is an AD-space and Y an FK-space.

Proposition 2.17: Let Y be any FK-space $\supset \Phi$. Then Y $\supset \Gamma_M$ if and only if the sequence $\delta^{(k)}$ is weakly analytic.

Proof: The following implications establish the result: since Γ_M has AD and by Lemma 2.16,

$$
\begin{aligned}
& Y \supset \Gamma_M \Leftrightarrow Y^f \subset (\Gamma_M)^f \\
& \Leftrightarrow Y^f \subset \wedge \quad (\text{since } (\Gamma_M)^f = \wedge) \\
& \Leftrightarrow \text{for each } f \in Y', \text{ the topological dual of Y. } f\left(\delta^{(k)}\right) \in \wedge \qquad \dots (2.47) \\
& \Leftrightarrow f\left(\delta^{(k)}\right) \text{ is analytic} \\
& \Leftrightarrow \delta^{(k)} \text{ is weakly analytic,}
\end{aligned}
$$

This completes the proof.

REFERENCES

1. Brown HL. The summability field of a perfect l-l method of summation. J Anal Math. 1967; 20:281–287.
2. Kamthan PK, Gupta M. Sequence spaces and series. Lecture Notes in Pure and Applied Mathematics. Vol. 65. New York: Marcel Dekker; 1981.
3. Lindenstrauss J, Tzafriri L. On Orlicz sequence spaces. Israel J Math. 1971; 10:379–390.
4. Maddox IJ. Sequence spaces defined by a modulus. Math Proc Cambridge Philos Soc. 1986;100(1):161–166.
5. Ruckle WH. FK spaces in which the sequence of coordinate vectors is bounded. Can J Math. 1973; 25:973–978.
6. Wilansky A. Summability through functional analysis. North-Holland Mathematics Studies. Vol. 85. Amsterdam: North-Holland Publishing; 1984.

About the corresponding author

Naveen Kumar Srivastava is affiliated with St. Andrew's College, Gorakhpur, Uttar Pradesh, India. He is actively engaged in teaching and academic activities, contributing to higher education. His areas of interest include research, student development, and academic excellence. He is committed to fostering learning, innovation, and intellectual growth among students.