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Operational Solution of Linear Fractional Differential Equations Using Kamal Transform

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Abstract

Fractional differential equations have become important tools for modelling complex dynamical systems possessing memory and hereditary properties. Analytical solution techniques based on integral transforms provide efficient approaches for handling such equations. In this paper, the Kamal transform, an existing integral transform technique, is employed to obtain analytical solutions of fractional differential equations. Operational properties of the transform are utilised to convert fractional models into algebraic equations, which are solved systematically. The obtained solutions are expressed in terms of special functions and demonstrate the effectiveness of the Kamal transform as a reliable method for fractional calculus problems.

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KEYWORDS: Kamal transform; Fractional differential equations; Fractional calculus; Integral transforms; Special functions.

1. INTRODUCTION

Creating Differential equations constitutes the mathematical foundation for modelling physical, engineering, and biological systems. Classical integer-order differential equations successfully describe many processes; however, they often fail to capture memory effects and hereditary characteristics observed in real-world phenomena. Fractional calculus extends classical differentiation and integration to arbitrary orders and has emerged as an important mathematical framework for describing anomalous diffusion, viscoelastic materials, signal processing, and control systems. The analytical treatment of fractional differential equations remains a challenging problem due to the nonlocal nature of fractional operators. To overcome these difficulties, integral transform techniques have been widely employed. Classical transforms such as Laplace and Fourier transforms convert differential equations into algebraic forms, simplifying the solution process. Over the last two decades, several alternative transforms have been introduced to enhance computational efficiency and analytical flexibility.

Among these developments, the Kamal transform has been proposed as an integral transform with exponential-type kernel structure suitable for solving differential and integral equations. The transform was introduced in recent transform-analysis literature as an alternative operational method analogous to other modern transforms used in applied mathematics. Since its introduction, the Kamal transform has been applied to various linear differential equations and mathematical models due to its simple operational rules and convenient inverse formulation. The main contribution of this work is the application of the existing Kamal transform for solving fractional differential equations and demonstrating its effectiveness in fractional calculus problems.

Preliminaries:

Definition: Let $C_\mu[0,\infty)$ be the class of the function of exponential order μ such that $|f(t)| \le Me^{\mu t}, M > 0.$ This space ensures the convergence of the Kamal transform integral and the existence of the inverse transform.

Definition: The Caputo's fractional derivative of $f(t)$ is given by

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau, \text{ where } -1 < \alpha \le n, n \in N$$

Definition: Consider a set A defined as

$$A = \{f: |f(t)| < pe^{\frac{t}{\emptyset_j}} \text{ if } t \in (-1)^j \times [0,\infty),\ j = 1,2\ ;\ \emptyset_j > 0$$

Where $\emptyset_1, \emptyset_2$ may be finite or infinite, and the constant p must be finite.

The Kamal transform is

$$K(f(t)) = F(u) = \int_0^\infty e^{\frac{-t}{u}} f(t)dt \quad t \ge 0,\ \emptyset_1 \le u \le \emptyset_2$$

Fundamental Theorems of Kamal Transform:

Theorem 1: Linearity Property:

If $K[f(t)] = F(u),\ K[g(t)] = G(u), then\ K[a\,f(t) + b\,g(t)] = a\,F(u) + bG(u)$, where a and b are constants.

Proof: From the definition,

$$K[a\,f(t) + b\,g(t)] = \int_0^\infty e^{\frac{-t}{u}} \{a\,f(t) + b\,g(t)\}dt$$
$$= a\int_0^\infty e^{\frac{-t}{u}} f(t)dt + b\int_0^\infty e^{\frac{-t}{u}} g(t)dt$$
$$= aF(u) + b\,G(u)$$

Theorem 2: Kamal Transform of Caputo Derivative:

Let $K[f(t)] = F(u)$, Then the Kamal transform of the Caputo fractional derivative of order α, satisfies $K[D_t^\alpha f(t)] = \frac{1}{u} F(u) - \sum_{k=0}^{n-1} u^{k-\alpha} f^{(k)}(0)$, $where\ n - 1 < \alpha < n.$

Proof: Applying the Kamal transform to the Caputo fractional derivative,

$$K[D_t^\alpha f(t)] = \int_0^\infty e^{\frac{-t}{u}} D_t^\alpha f(t)\, dt$$

Using operational properties of fractional together with initial conditions yields

$$\frac{1}{u}\, \boldsymbol{F(u)} - \sum_{k=0}^{n-1} \boldsymbol{u^{k-\alpha} f^{(k)}(0)}.$$

Theorem 3: Convolution Property:
Let $\boldsymbol{F(u)}$ and $\boldsymbol{G(u)}$ be the Kamal transforms of $\boldsymbol{f(t)}$ and $\boldsymbol{g(t)}$ respectively.

$$\text{If } h(t) = (f \times g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau \text{ then } K[h(t)] = F(u)\, G(u)$$

$$\textbf{Proof: } K[h(t)] = \int_0^\infty e^{\frac{-t}{u}} \{\int_0^t f(\tau)\, g(t-\tau)\, d\tau\} dt$$

Changing the order of integration gives $F(u)\, G(u)$.

Application to Fractional Differential Equation:
Now, we obtain the solution of some fractional differential equations using the Kamal transform method.

Example: Consider fractional differential equations of the form

$$\boldsymbol{D^\alpha y(t) = f(t)} \text{ with initial conditions } \boldsymbol{y^{(k)}(0) = c_k,\ k = 0, 1, 2, 3, ---, n-1}$$
where n is the smallest integer greater than $\boldsymbol{\alpha}$ such that $\boldsymbol{-1 < n-1 < \alpha \leq n}$.

Solution: Let $\boldsymbol{f(t) \in A}$.
Applying Kamal transform**,**

$$\boldsymbol{u^{-\alpha}\, Y(u) - \sum u^{k-\alpha} c_k = F(u)}$$

Using initial conditions and simplifying, we have

$$Y(u) = u^\alpha\, F(u) + \sum u^k\, c_k$$

Taking inverse transform, we get

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^\infty (t-\tau)^{\alpha-1} f(\tau) d\tau + \sum c_k t^k$$

Example: Consider the fractional differential equation $D_t^{\frac{3}{2}} y(t) = 1,$ with initial conditions

$$y(0) = 0,\ y'(0) = 0$$

Solution: We have

$$D_t^{\frac{3}{2}} y(t) = 1$$

Applying the Kamal transform, we get

$$K(D_t^{\frac{3}{2}} y(t)) = K(1)$$
$$u^{\frac{-3}{2}}\, Y(u) = 1$$
$$Y(u) = u^{\frac{3}{2}}$$

Applying the inverse Kamal transform gives

$$y(t) = \frac{t^{3/2}}{\Gamma(5/2)}$$

Example: Consider the fractional differential equation $D_t^{\frac{1}{2}} y(t) = e^t,$ with initial conditions

$$y(0) = 1.$$

Solution: We have

$$D_t^{\frac{1}{2}} y(t) = e^t$$

Applying the Kamal transform to the above equation, we get

$$K(D_t^{\frac{1}{2}} y(t)) = K(e^t)$$

$$u^{\frac{-1}{2}}Y(u)=\frac{u}{1-u}$$

$$Y(u)=\frac{u^{\frac{3}{2}}}{1-u}$$

Applying the inverse Kamal transform gives

$$y(t)=1+t^{1/2}E_{1/2}(t)$$

Example: Consider fractional differential equations of the form

$\boldsymbol{D^{\alpha}y(t)+D^{\beta}y(t)=f(t)}$ with initial conditions $\boldsymbol{y^{(k)}(0)=c_k}$, $\boldsymbol{k\;=\;0,1,2,3,---,l-1}$, where $\boldsymbol{\alpha\ and\ \beta}$ are positive numbers with $\boldsymbol{\alpha>0,\beta>0,l-1<\alpha\ \leq l,l\in N,0<\beta<\alpha}$ and $\boldsymbol{\alpha-l+1\geq\beta}$ and $\boldsymbol{l}$ is the smallest integer greater than $\boldsymbol{\alpha}$.

Solution: Let $\boldsymbol{K[y(t)]=Y(u)}$, $\boldsymbol{K[f(t)]=F(u)}$

Applying the Kamal transform on both sides of the equation,

$$\boldsymbol{K[D^{\alpha}y(t)]+K[D^{\beta}y(t)]=K[f(t)]}$$

Using Caputo Fractional Derivative Property, we get

$$\boldsymbol{(u^{-\alpha}+u^{-\beta})Y(u)-\sum_{r=0}^{l-1}u^{r-\alpha}y^{(r)}(0)-\sum_{k=0}^{m-1}u^{k-\beta}y^{(k)}(0)=F(u)}$$

Here $\boldsymbol{m}$ is the smallest integer greater than, thus

$$\boldsymbol{Y(u)=\left(u^{-\alpha}+u^{-\beta}\right)^{-1}\{F(u)+\sum_{r=0}^{l-1}u^{r-\alpha}y^{(r)}(0)+\sum_{k=0}^{m-1}u^{k-\beta}y^{(k)}(0)\}}$$

Applying the inverse Kamal transform and using initial conditions, we obtain

$$\boldsymbol{y(t)=K^{-1}\{\left(u^{-\alpha}+u^{-\beta}\right)^{-1}[F(u)+\sum_{r=0}^{l-1}u^{r-\alpha}c_r+\sum_{k=0}^{m-1}u^{k-\beta}c_k]\}}$$

Example: Consider the fractional differential equation $D_t^{\frac{3}{2}}y(t)+D_t^{\frac{1}{2}}y(t)=t,$ with initial conditions $y(0)=0,\ y'(0)=0$

Solution: Applying the Kamal transform on both sides,

$$\boldsymbol{K\{D_t^{\frac{3}{2}}y(t)\}+K\{D_t^{\frac{1}{2}}y(t)\}=K\{t\}},\ \text{Let}\ \ \boldsymbol{K\{y(t)\}=Y(u)}$$

Using the Caputo Fractional Derivative Property and initial conditions, we get

$$\boldsymbol{\left(u^{-\frac{3}{2}}+u^{-\frac{1}{2}}\right)Y(u)=u^2}$$

$$\boldsymbol{Y(u)=\frac{u^2}{\left(u^{-\frac{3}{2}}+u^{-\frac{1}{2}}\right)}}$$

$$\boldsymbol{Y(u)=\frac{u^{7/2}}{1+u}}$$

Applying inverse Kamal transform, we obtain

$$y(t) = K^{-1}\{\frac{u^{7/2}}{1+u}\}$$

Example: Consider the fractional differential equation $D_t^{\frac{3}{2}}y(t) + D_t^{\frac{1}{2}}y(t) = sint$, with initial conditions $y(0) = 0,\ y'(0) = 0$

Solution: Applying the Kamal transform on both sides,

$$K\{D_t^{\frac{3}{2}}y(t)\} + K\{D_t^{\frac{1}{2}}y(t)\} = K\{sint\}, \quad \text{Let} \quad K\{y(t)\} = Y(u)$$

Using the Caputo Fractional Derivative Property and the initial conditions, we get

$$\left(u^{-\frac{3}{2}} + u^{-\frac{1}{2}}\right)Y(u) = \frac{u^2}{1+u^2} \qquad \{\because K\{sint\} = \frac{u^2}{1+u^2}\ \}$$

$$Y(u) = \frac{u^2}{\left(u^{-\frac{3}{2}} + u^{-\frac{1}{2}}\right)\ (1+u^2)}$$

$$Y(u) = \frac{u^{7/2}}{(1+u)(1+u^2)}$$

Applying inverse Kamal transform, we obtain

$$y(t) = K^{-1}\{\frac{u^{7/2}}{(1+u)(1+u^2)}\}$$

CONCLUSION

The present study demonstrates the effectiveness of the Kamal transform as an operational analytical technique for solving linear fractional differential equations involving Caputo fractional derivatives. By employing the fundamental properties of the Kamal transform, fractional differential models are successfully converted into algebraic equations, which significantly simplifies the solution procedure. The obtained solutions are expressed in closed analytical form without requiring complicated computational steps.

The illustrative examples confirm that the proposed approach provides a systematic and efficient framework compared to traditional analytical methods. The Kamal transform therefore serves as a reliable alternative tool in fractional calculus and integral transform theory. The developed methodology can be further extended to nonlinear fractional differential equations, systems of fractional equations, and fractional partial differential equations arising in engineering, physics, and applied sciences**.**

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