Research Article

Fixed Point Results Using New Helping Functions via Caristi Type

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ABSTRACT

Deviating from the traditional framework to prove the existence and uniqueness of a fixed point and replacing the fixed number in the Banach contraction principle with a function that has its conditions is one of the most difficult challenges facing studies concerned with the fixed point, which researchers took on recently. Success in such studies has a wide applied impact in many areas of mathematics, reflecting positively on various applied sciences. In this study, we establish new fixed point theorems for contractive mapping in a complete metric space using some helping functions via Caristi-type.

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1. INTRODUCTION

Banach contraction principle is a fundamental theorem in fixed-point theory. Due to the wide range of applications of fixed point theory, the Banach principle has undergone many extensions and generalizations, see [1,7,9,10]. The most notable of these extensions is what Caristi introduced in 1976 in [4,5] he employed a helping function to proof the exist and uniqueness. Du [6] has obtained a helping function (semi-lower continuous function) to support Caristi's fixed-point theorem. The fixed point theorem by Caristi's type is the subject of numerous recent publications, among these recent studies are [2,3,11]. This paper employs contractive mapping in the Banach metric space using a new helping function to obtain some new conclusions along the lines of the Caristi type. This paper is inspired by some new work on the extension of the Banach contraction principle.

Definition 1.1 [6] A semi lower continuous function ξ at u_0, in a metric space U can be expressed as for each

$$u, u_0 \in U \text{ then } \lim_{u \to u_0} sup\, \xi u \le \xi u_0.$$

Theorem 1.1 [12] Let (U, σ) be a complete metric space and ξ satisfies Definition 1.1. Suppose that ζ is a Caristi type mapping on U dominated by ξ; that is, for each $u \in U$ satisfies

$$\sigma(u, \zeta u) \le \xi u - \xi(\zeta u).. \tag{1.1}$$

Then ζ has a fixed point in U.

In this study, we derive new fixed-point theorems for contractive mapping in a complete metric space via the Caristi type, using new helping functions.

2. MAIN RESULTS.

Theorem 2.1 Let $\zeta: U \to U$ be a contractive mapping on a complete metric space(U, σ). Suppose there exists a function $\eta: U \to (0, +\infty)$ such that

$$\eta u = \sigma(\zeta u, u), \tag{2.1}$$

For all $u \in U$, satisfies

$$\sigma(\zeta u, \zeta v) \le (\eta u + \eta v) - \big(\eta(\zeta u) + \eta(\zeta v)\big), \tag{2.2}$$

For all $u, v \in U$ and $u \ne v$. Then ζ has a unique fixed point.

Proof. For all $n \in \mathbb{N}$, consider

$$u_{n+1} = \zeta u_n. \tag{2.3}$$

Now,

$$\sigma(u_n, u_{n+1}) = \sigma(\zeta u_{n-1}, \zeta u_n) \le (\eta u_{n-1} + \eta u_n) - (\eta \zeta u_{n-1} + \eta \zeta u_n)) \tag{2.4}$$

$$= \eta u_{n-1} - \eta u_{n+1}.$$

Using (2.3),

$$\sigma(u_n, u_{n+1}) \le \sigma(\zeta u_{n-1}, u_{n-1}) - \sigma(\zeta u_{n+1}, u_{n+1})$$
$$\sigma(u_n, u_{n+1}) + \sigma(u_{n+1}, u_{n+2}) \le \sigma(u_{n-1}, u_n). \tag{2.5}$$

And,

$$\sigma(u_n, u_{n+2}) < \sigma(u_{n-1}, u_n). \tag{2.6}$$

From [2], ηu is continuous and limited down bounded indeed ζ is contractive. Then from (2.6) we conclude that u_n is decreasing to some point in U, so there exists $\omega \in U$ such that for all $u \in U$,

$$\eta \omega \le \eta u. \tag{2.7}$$

Since, U is a complete space. So, $\{u_n\}$ is a convergent sequence. Let $u_n \to \omega$, $n \to \infty$, for all $n \in \mathbb{N}$. Then$u_{n_k} \to \omega$, $\{u_{n_k}\} \subseteq \{u_n\}$.

Now, we will proof the existence of fixed point of ζ in U. clearly, ζ is continuous, since ζ is contractive [8], then

$$\zeta\left(\lim_{n \to \infty} u_{n_k}\right) = \zeta \omega, \; k \to \infty. \tag{2.8}$$

Therefore ω is a fixed point of ζ in U.

To proof the uniqueness of a fixed point suppose there is another point $\varpi \in U$ such that $\omega \ne \varpi$ and $\zeta \varpi = \varpi$, for all $\omega, \varpi \in U$.

Therefore,

$$\sigma(\omega, \varpi) = \sigma(\zeta \omega, \zeta \varpi) \le (\eta \omega + \eta \varpi) - \big(\eta(\zeta \omega) + \eta(\zeta \varpi)\big). \tag{2.9}$$
$$= (\eta \omega + \eta \varpi) - (\eta \omega + \eta \varpi).$$

Hence, $\omega = \varpi$ and there is a unique fixed point of ζ.

Theorem 2.2 In theorem 2.1, if for some $\iota \in N$ and ζ^ι is a contractive. Then ζ has a unique fixed point.

Proof. Since ζ^ι is contractive, then ζ^ι is continuous. So, in a similar proof of theorem 2.1, ζ^ι has a unique fixed point. Suppose that the fixed point is $\omega \in U$. We need to prove that $\omega \in U$ is a unique fixed point of ζ. For that, we assume $\zeta^r \omega \ne \omega$, $\forall\, r = 1,2, \ldots, \iota - 1$.

Let, for all $\omega \ne \varpi \in U$

$$\zeta^r \omega = \varpi \tag{2.10}$$

Then,

$$\sigma(\zeta^r \omega, \omega) = \sigma(\zeta^r \omega, \zeta^r \omega) = \sigma(\zeta(\zeta^{r-1}\omega, \zeta(\zeta^{\iota-1}\omega)). \tag{2.11}$$

So, by (2.1),

$$\sigma(\zeta^r \omega, \omega) \le \big(\eta(\zeta^{r-1}\omega) + \eta(\zeta^{m-1}\omega)\big) - (\eta(\zeta(\zeta^{r-1}\omega) + \eta(\zeta(\zeta^{\iota-1}\omega)).$$
$$= \eta(\zeta^{r-1}\omega) + \eta(\zeta^{\iota-1}\omega) - \eta(\zeta^r \omega) - \eta(\zeta^\iota \omega).$$
$$= 0.$$

By (2.10) and $\iota > 1$. this implies that $\zeta^r \omega = \omega \quad \forall\, r = 1,2, \ldots, \iota - 1$.

Hence, ω is a fixed point of ζ. now, we will prove that ω is a unique fixed point. So, suppose $w \in U$ is such that $w \ne \omega$ and $\zeta w = w$. Therefore,

$\sigma(\omega, w) = \sigma(\zeta\omega, \zeta w) \leq (\eta\omega + \eta w) - (\eta(\zeta\omega) + \eta(\zeta w)) = 0.$

Theorem 2.3 Let $\zeta: U \to U$ be a contractive mapping on a complete metric space U and there exists a function $\eta: U \to (0, \infty)$ such that (2.3) is satisfied. If,

$$\sigma(\zeta^m u, \zeta^n v) \leq (\eta u + \varphi v) - \left(\eta(\zeta^m u) + \eta(\zeta^n v)\right), \quad \textbf{(2.12)}$$

For all $u, v \in U$, $u \neq v$, and $m, n \in N$, then ζ has a unique fixed point in U.

Proof. Since this theorem's proof is similar to earlier theorems, it has been omitted.

Example 2.1 Consider $\zeta u = u^2$ defined on $U = [0,1]$ and take $\eta u: (-\infty, \infty) \to (0, \infty)$ be a lower semi continuous function defined by

$$\eta u = \begin{cases} 0, & u \leq 0 \\ 1, & u > 1 \end{cases} \quad \textbf{(2.13)}$$

So, for all $u, v \in U$, such that $u \neq v$.

We have,

$$\begin{aligned} \sigma(\zeta u, \zeta v) &= \sigma(u^2, v^2) \\ &= (u + u)\, \sigma(u, u). \\ &< \sigma(u, u). \end{aligned}$$

Thus, for ζ is contractive mapping and by (2.1), (2.3) we get,

$$\begin{aligned} \sigma(\zeta u, \zeta v) &\leq (\eta u + \eta v) - (\eta(\zeta u) + \eta(\zeta v). \\ &= \sigma(\zeta u, u) + \sigma(\zeta v, v) - [\sigma(\zeta(\zeta u, \zeta u) + \sigma(\zeta(\zeta v, \zeta v)]. \\ &= \sigma(u^2, u) + \sigma(v^2, v) - \sigma(u^4, u^2) - \sigma(v^4, v^2). \\ &= \sigma(u^2, u) - \sigma(u^4, u^2) + \sigma(v^2, v) - \sigma(v^4, v^2). \\ &= \sigma(u, u^4) + \sigma(v, v^4). \end{aligned} \quad \textbf{(2.14)}$$

So,

$$\sigma(\zeta u, \zeta v) \leq \sigma(u, \zeta^2 u) + (v, \zeta^2 v). \quad \textbf{(2.15)}$$

We have two cases:

I. If $u \neq v$ then (3.5) is satisfied.

II. If $u = v$, then $\zeta u = \zeta v$ which implies that u is the fixed point of ζ. Hence, (2.15) is also satisfied.

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