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Domination Number and Total Domination Number of Square of Normal Product of Paths and Cycles

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Abstract

In this chapter, we propose a method, to find the dominating set and total dom- inating set of square of $P_m \boxtimes P_n$ and $C_m \boxtimes C_n$. This approach relies on the cardinal- ity of the minimum dominating set and minimum total dominating set of graph G. Moreover, a few illustrations and their short display are designed to understand the outcomes.

Keywords: Reference management software, Scopus, Bibliometric study, Citation.

Introduction

Henning (2009) studied a few leading outcomes related to the algorithms of total dominating set. Further, some crucial outcomes in respect of the total dominating sets of different graphs were provided by Atapour and Soltankhan (2009). In this sequence, Alishahi and Shalmaee (2015) proposed square of graphs and deduced an important result for γ(Pm2Pn) and γ(Cm2Cn). The dominating set of normal product of paths and cycles was identified by Chaluvaraju and Appajigowda (2015). They further described so many different outcomes regarding the normal product of paths and its dominating sets.

Walikar proposed the idea related to the square graphs during his lessons. Ball and Coxeter (1987) illustrated that for any graph G, its square graph G2 can be a graph with the same vertex set V (G) and two vertices would be adjacent whenever they are at distance 1 or 2 in graph G. In this chapter, Section 3.2 and 3.3 contains the results on the domination number of (Pm ⊠ Pn)2 and (Cm ⊠ Cn)2. Further, total domination number of (Pm ⊠ Pn)2 and (Cm ⊠ Cn)2 have been derived in section 3.4.

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3.2 Domination Number of $(P_m \square P_n)^2$

In this section, we determine the domination number of the square of the Cartesian (normal) product of path graphs for $m = 1,2,$ ...by identifying a minimum dominating set.

Theorem 3.2.1

Let $P_n$be a path graph with $n \geq 2$vertices. Then the domination number of the square of the Cartesian product $P_3 \square P_n$is

$$\gamma[(P_3 \square P_n)^2] = \left\lceil \frac{n}{5} \right\rceil.$$

Proof

Let P_3be a path graph with vertex set

$$V(P_3) = \{u_1, u_2, u_3\},$$

and let $P_n$be a path graph with vertex set

$$V(P_n) = \{v_1, v_2, \dots, v_n\}.$$

The vertex set of the Cartesian product $P_3 \square P_n$is

$$V(P_3 \square P_n) = \{(u_i, v_j) \mid 1 \le i \le 3,\ 1 \le j \le n\}.$$

Two vertices $(u_i, v_j)$and $(u_r, v_s)$are adjacent in $P_3 \square P_n$if and only if

- $i = r$and $v_j v_s \in E(P_n)$, or
- $j = s$and $u_i u_r \in E(P_3)$.

Now consider the dominating set

$$D = \{(u_3, v_{5k-2}) \mid 1 \le k \le \left\lceil \frac{n}{5} \right\rceil\}.$$

Clearly,

$$\mid D \mid = \left\lceil \frac{n}{5} \right\rceil.$$

In the square graph $(P_3 \square P_n)^2$, every vertex within distance at most 2 from any vertex in $D$is dominated. Each vertex (u_3, v_{5k-2})dominates all vertices in the columns v_{5k-4}through v_{5k}, across all three rows of P_3. Hence, the entire vertex set of $(P_3 \square P_n)^2$is dominated by D.

Furthermore, no dominating set of smaller cardinality can exist, since fewer vertices would leave at least one column of vertices undominated.

Thus, $D$is a minimum dominating set of $(P_3 \square P_n)^2$, and therefore,

$$\gamma[(P_3 \square P_n)^2] = \left\lceil \frac{n}{5} \right\rceil.$$

This result can be seen in Figure 3.1.

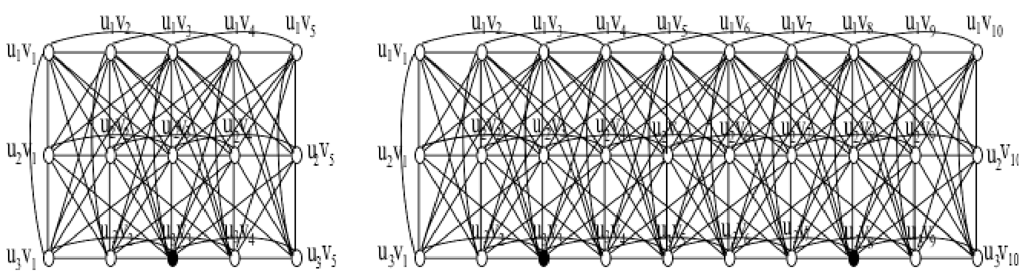

Figure 3.1: A dominating set for (P3 ⊠ P5)2 and (P3 ⊠ P10)2

Theorem 3.2.2

If $P_m \square P_n$is the normal product of path graphs $P_m$and $P_n$having $m$and $n$vertices wherein $m, n \ge 2$, then the domination number of $(P_m \square P_n)^2$is

$$\gamma[(P_m \square P_n)]^2 = k_1 k_2,$$

where

$$k_1 = \left\lceil \frac{m}{5} \right\rceil \text{and} k_2 = \left\lceil \frac{n}{5} \right\rceil.$$

Proof:

Suppose that $P_m$is a graph having $m$vertices namely $u_1, u_2, \dots, u_m$and $P_n$is a graph having $n$vertices namely $v_1, v_2, \dots, v_n$. Denote

$$S = \{(u_i, v_j) \in V(P_m \square P_n) \text{ such that } i = 1,2,\dots,m \text{ and } j = 1,2,3,\dots,n\}$$

as vertex set of $(P_m \square P_n)$and $(u_i, v_j)$and $(u_r, v_r)$are adjacent if $u_i = u_r$or $v_j = v_r$and if $u_i = u_r$and

$$v_j v_r \in E(P_n)$$

or $v_j = v_r$and

$$u_i u_r \in E(P_m).$$

According to the values of $m$and n, following two cases arise.

Case 1:

For

$$m = 5k_1 - 2 \text{or} 5k_1 - 1 \text{or} 5k_1, k_1 \ge 1$$

In this case, we consider two sub cases as follows.

Subcase 1:
Take

$$n = 5k_2 - 2 \text{or} 5k_2 - 1 \text{or} 5k_2, k_2 \geq 1,$$

In $[(P_m \square P_n)]^2$, the subset

$$A = \{(u_{5t_1-2}, v_{5t_2-2});\ 1 \leq t_1 \leq k_1, 1 \leq t_2 \leq k_2\}$$

is the dominating set. It is clear that there is no dominating set with cardinality less than

$$| A |$$

in G. Thus, $A$is the minimum dominating set of $[(P_m \square P_n)]^2$. Hence,

$$\gamma[(P_m \square P_n)]^2 = | A | = k_1 k_2.$$

Subcase 2:
Let

$$n = 5k_2 - 4 \text{or} 5k_2 - 3, k_2 \geq 1,$$

In $[(P_m \square P_n)]^2$, the subset

$$B = \{(u_{5t_1-3}, v_{5t_2-4});\ 1 \leq t_1 \leq k_1, 1 \leq t_2 \leq k_2\}$$

is the dominating set. Clearly, there exists no dominating set having cardinality less than

$$| B |$$

in the graph G. Thus, $B$is the minimum dominating set of $[(P_m \square P_n)]^2$. Therefore,

$$\gamma[(P_m \square P_n)]^2 = | B | = k_1 k_2.$$

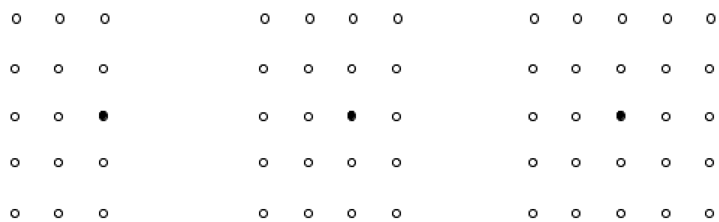

Figure 3.2: A dominating set for $(P_5 \boxtimes P_3)^2$, $(P_5 \boxtimes P_4)^2$ and $(P_5 \boxtimes P_5)^2$ (Subcase 1)

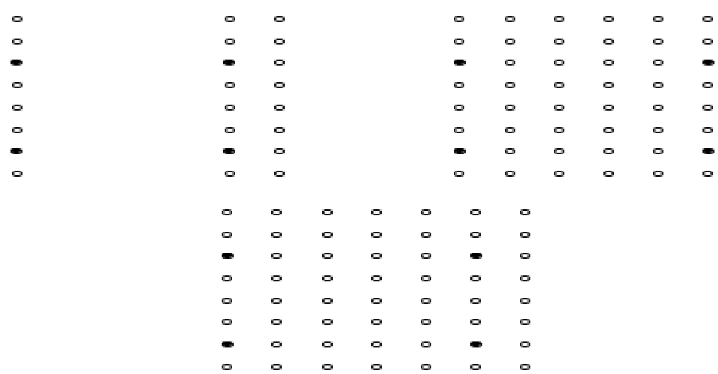

Figure 3.3: A dominating set for $(P_8 \boxtimes P_1)^2$, $(P_8 \boxtimes P_2)^2$, $(P_8 \boxtimes P_6)^2$ and $(P_8 \boxtimes P_7)^2$(Subcase 2)

Case 2:
Take

$$m = 5k_1 - 3 \text{or} 5k_1 - 4, k_1 \geq 2$$

In this case, we consider the two subcases as follows.

Subcase 1:
Let

$$n = 5k_2 - 2 \text{or} 5k_2 - 1 \text{or} 5k_2, k_2 \geq 1,$$

In $[(P_m \square P_n)]^2$, the subset

$$C = \{(u_{5t_1-3}, v_{5t_2-2});\ 1 \leq t_1 \leq k_1, 1 \leq t_2 \leq k_2\}$$

is the dominating set. Clearly, there is no dominating set with cardinality less than

$$|C|$$

in G. Hence, $C$is the minimum dominating set of $[(P_m \square P_n)]^2$. Therefore,

$$\gamma[(P_m \square P_n)]^2 = |C| = k_1 k_2.$$

Subcase 2:

Let

$$n = 5k_2 - 4 \text{or} 5k_2 - 3, k_2 \geq 2,$$

In $[(P_m \square P_n)]^2$, the subset

$$D = \{(u_{5t_1-3}, v_{5t_2-4});\ 1 \leq t_1 \leq k_1, 1 \leq t_2 \leq k_2\}$$

is the dominating set. Clearly, there is no dominating set with cardinality less than

$$|D|.$$

Thus, $D$is the minimum dominating set of $[(P_m \square P_n)]^2$. Hence,

$$\gamma[(P_m \square P_n)]^2 = |D| = k_1 k_2.$$

In Case 1 and Case 2, it is clear that

$$\gamma[(P_m \square P_n)]^2 = k_1 k_2$$

where

$$k_1 = \left\lceil \frac{m}{5} \right\rceil \text{and} k_2 = \left\lceil \frac{n}{5} \right\rceil.$$

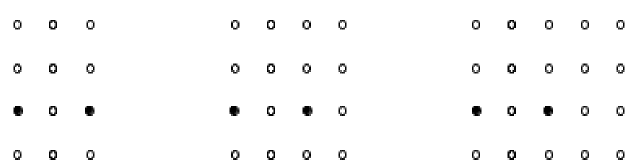

Figure 3.4: A dominating set for $(P_4 \boxtimes P_3)^2$, $(P_4 \boxtimes P_4)^2$ and $(P_4 \boxtimes P_5)^2$ (Subcase 1)

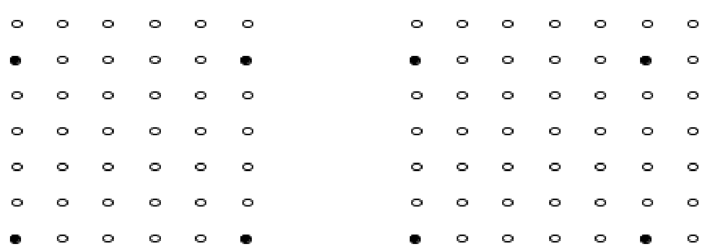

Figure 3.5: A dominating set for $(P_7 \boxtimes P_6)^2$ and $(P_7 \boxtimes P_7)^2$ (Subcase 2)

Domination Number of $(C_m \square C_n)^2$

In this section, we investigate the domination number of a square of normal product of two cycles $C_m$and $C_n$for $m, n \geq 3$.

Theorem 3.3.1

Let $C_m \square C_n$be the normal product of cycle graphs $C_m$and $C_n$having $m$and $n$vertices, wherein $m, n \geq 3$. For

$$m = 5k_1 - 2 \text{or} 5k_1 - 1 \text{or} 5k_1, k_1 \geq 1$$

and

$$m = 5k_1 - 3, k_1 \geq 2,$$

the domination number of $(C_m \square C_n)^2$is

$$\gamma[(C_m \square C_n)]^2 = k_1 k_2,$$

where

$$k_1 = \left\lceil \frac{m}{5} \right\rceil \text{and} k_2 = \left\lceil \frac{n}{5} \right\rceil.$$

Proof :

Suppose that $C_m$is a graph having $m$vertices namely $u_1, u_2, \dots, u_m$and $C_n$is a graph having $n$vertices namely $v_1, v_2, \dots, v_n$. Denote

$$S = \{(u_i, v_j) \in V(C_m \square C_n) \text{ such that } i = 1,2, \dots, m \text{ and } j = 1,2,3, \dots, n\}$$

as vertex set of $(C_m \square C_n)$and $(u_i, v_j)$and $(u_r, v_r)$are adjacent if $u_i = u_r$or $v_j = v_r$and if $u_i = u_r$and

$$v_j v_r \in E(C_n)$$

or $v_j = v_r$and

$$u_i u_r \in E(C_m).$$

According to the values of $m$and n, the following two cases arise.

Case 1:

Let

$$m = 5k_1 - 2 \text{or} 5k_1 - 1 \text{or} 5k_1, k_1 \geq 1.$$

In this case, we consider the two subcases as follows:

Subcase 1:

Let

$$n = 5k_2 - 2 \text{or} 5k_2 - 1 \text{or} 5k_2, k_2 \geq 1.$$

In $[(C_m \square C_n)]^2$, the subset

$$A = \{(u_{5t_1-2}, v_{5t_2-2});\ 1 \leq t_1 \leq k_1, 1 \leq t_2 \leq k_2\}$$

is the dominating set. Clearly, there is no dominating set with cardinality less than

$$|A|$$

in the graph. Thus, $A$is the minimum dominating set of $[(C_m \square C_n)]^2$. Hence,

$$\gamma[(C_m \square C_n)]^2 = |A| = k_1 k_2.$$

Subcase 2:

Let

$$n = 5k_2 - 4 \text{or} 5k_2 - 3, k_2 \geq 1.$$

In $[(C_m \square C_n)]^2$, the subset

$$B = \{(u_{5t_1-3}, v_{5t_2-4});\ 1 \leq t_1 \leq k_1, 1 \leq t_2 \leq k_2\}$$

is the dominating set. Clearly, no dominating set with cardinality less than

$$|B|$$

in the graph. Thus, $B$is the minimum dominating set of $[(C_m \square C_n)]^2$. Therefore,

$$\gamma[(C_m \square C_n)]^2 = |B| = k_1 k_2.$$

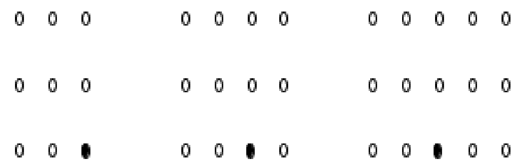

Figure 3.6: A dominating set for $(C_3 \boxtimes C_3)^2$, $(C_3 \boxtimes C_4)^2$ and $(C_3 \boxtimes C_5)^2$ (Subcase 1)

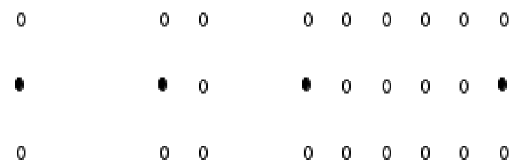

Figure 3.7: A dominating set for $(C_3 \boxtimes C_1)^2$, $(C_3 \boxtimes C_2)^2$ and $(C_3 \boxtimes C_6)^2$ (Subcase 2)

Case 2:
Let

$$m = 5k_1 - 3, k_1 \geq 2.$$

In this case, we consider the following two subcases :

Subcase 1:
Let

$$n = 5k_2 - 2 \text{or} 5k_2 - 1 \text{or} 5k_2, k_2 \geq 1.$$

In $[(C_m \square C_n)]^2$, the subset

$$C = \{(u_{5t_1-3}, v_{5t_2-2});\ 1 \leq t_1 \leq k_1, 1 \leq t_2 \leq k_2\}$$

is the dominating set. Clearly, there is no dominating set with cardinality less than

$$|\ C\ |$$

in the graph. Thus, $C$is the minimum dominating set of $[(C_m \square C_n)]^2$. Hence,

$$\gamma[(C_m \square C_n)]^2 = |\ C\ | = k_1 k_2.$$

Subcase 2:
Let

$$n = 5k_2 - 4 \text{or} 5k_2 - 3, k_2 \geq 1.$$

In $[(C_m \square C_n)]^2$, the subset

$$D = \{(u_{5t_1-3}, v_{5t_2-4});\ 1 \leq t_1 \leq k_1, 1 \leq t_2 \leq k_2\}$$

is the dominating set. Clearly, there is no dominating set with cardinality less than

$$|\ D\ |$$

in the graph. Thus, $D$is the minimum dominating set of $[(C_m \square C_n)]^2$. Therefore,

$$\gamma[(C_m \square C_n)]^2 = |\ D\ | = k_1 k_2.$$

In Case 1 and Case 2, it is clear that

$$\gamma[(C_m \square C_n)]^2 = k_1 k_2$$

for

$$m = 5k_1 - 2 \text{or} 5k_1 - 1 \text{or} 5k_1, k_1 \geq 1$$

and

$$m = 5k_1 - 3, k_1 \geq 2$$

where

$$k_1 = \left\lceil \frac{m}{5} \right\rceil \text{and} k_2 = \left\lceil \frac{n}{5} \right\rceil.$$

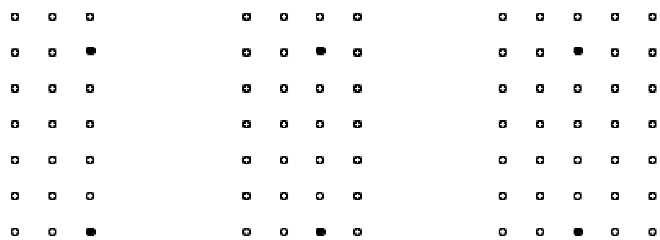

Figure 3.8: A dominating set for $(C_7 \boxtimes C_3)^2$, $(C_7 \boxtimes C_4)^2$ and $(C_7 \boxtimes C_5)^2$ (Subcase 1)

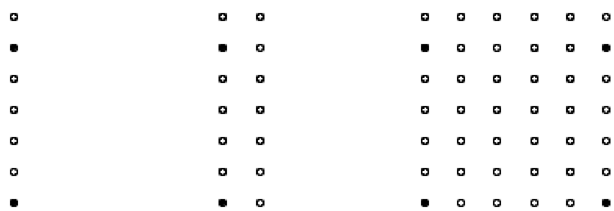

Figure 3.9: A dominating set for $(C_7 \boxtimes C_1)^2$, $(C_7 \boxtimes C_2)^2$ and $(C_7 \boxtimes C_6)^2$ (Subcase 2)

Theorem 3.3.2
For

$$m = 5k_1 - 4, k_1 \geq 2,$$
$$\gamma[(C_m \square C_n)]^2 = \begin{cases} k_1 k_2, & \text{if } n = 5k_2 - 3 \text{ or } 5k_2 - 2 \text{ or } 5k_2 - 1 \text{ or } 5k_2, k_2 \geq 1, \\ k_1 k_2 - 1, & \text{if } n = 5k_2 - 4, k_2 \geq 2. \end{cases}$$

where,

$$k_1 = \left\lceil \frac{m}{5} \right\rceil \text{and} k_2 = \left\lceil \frac{n}{5} \right\rceil.$$

Proof:
Suppose that $C_m$has vertices $u_1, u_2, \dots, u_m$and $C_n$has vertices $v_1, v_2, \dots, v_n$. Denote

$$S = \{(u_i, v_j) \in V(C_m \square C_n) \text{ such that } i = 1,2, \dots, m \text{ and } j = 1,2,3, \dots, n\}$$

as vertex set of $(C_m \square C_n)$and there is an edge between $(u_i, v_j)$and (u_r, v_r)if $u_i = u_r$or $v_j = v_r$and if $u_i = u_r$and

$$v_j v_r \in E(C_n)$$

or $v_j = v_r$and

$$u_i u_r \in E(C_m).$$

According to the values of $m$and n, the following three sub cases arise :

Subcase 1:
Let

$$n = 5k_2 - 3 \text{or} 5k_2 - 2, k_2 \geq 1.$$

In $[(C_m \square C_n)]^2$, the subset

$$A = \{(u_{5t_1-2}, v_{5t_2-4});\ 1 \le t_1 \le k_1, 1 \le t_2 \le k_2\}$$

is the dominating set. Definitely, there is no dominating set having cardinality less than

$$\mid A \mid$$

in the graph. Thus, $A$is the minimum dominating set of $[(C_m \Box C_n)]^2$. Hence,

$$\gamma[(C_m \Box C_n)]^2 = \mid A \mid = k_1 k_2.$$

Subcase 2:

Let

$$n = 5k_2 - 1 \text{or} 5k_2, k_2 \ge 1.$$

In $[(C_m \Box C_n)]^2$, the subset

$$B = \{(u_{5t_1-2}, v_{5t_2-2});\ 1 \le t_1 \le k_1, 1 \le t_2 \le k_2\}$$

is the dominating set. Surely, there is no dominating set with cardinality less than

$$\mid B \mid$$

in the graph. Thus, $B$is the smallest dominating set of $[(C_m \Box C_n)]^2$. Hence,

$$\gamma[(C_m \Box C_n)]^2 = \mid B \mid = k_1 k_2.$$

Subcase 3:

Let

$$n = 5k_2 - 4, k_2 \ge 2.$$

In $[(C_m \Box C_n)]^2$, the subset

$$\begin{aligned} C = & \quad \{(u_1, v_1)\} \\ & \cup \{(u_{5t_1+1}, v_{5t_2-2});\ 1 \le t_1 \le k_1 - 1, 1 \le t_2 \le k_2 - 1\} \\ & \cup \{(u_{5t_1-2}, v_{5t_2-4});\ 1 \le t_1 \le k_1 - 1, 1 \le t_2 \le k_2 - 1\} \end{aligned}$$

is the dominating set. Clearly, there is no dominating set with cardinality less than

$$\mid C \mid$$

in the graph. Thus, $C$is the minimum dominating set of $[(C_m \Box C_n)]^2$. Hence,

$$\gamma[(C_m \Box C_n)]^2 = \mid C \mid = k_1 k_2 - 1.$$

In all these three sub cases, we have the values of

$$k_1 = \left\lceil \frac{m}{5} \right\rceil \text{and} k_2 = \left\lceil \frac{n}{5} \right\rceil.$$

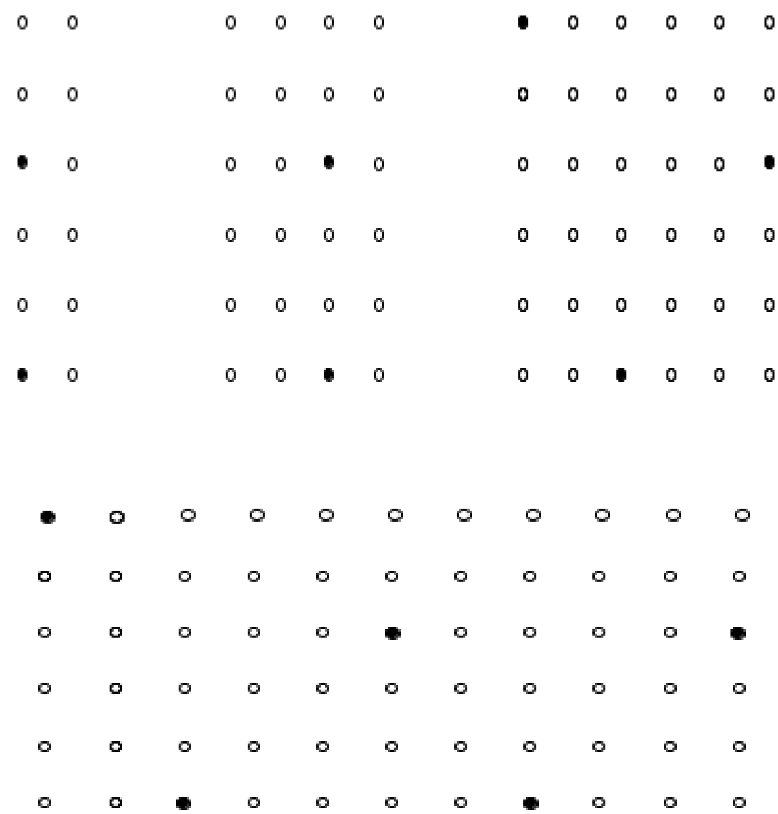

Figure 3.10: A dominating set for $(C_6 \boxtimes C_2)^2$, $(C_6 \boxtimes C_4)^2$, $(C_6 \boxtimes C_6)^2$ and $(C_6 \boxtimes C_{11})^2$

Total Domination Number of $(P_m \square P_n)^2$ and

$$(C_m \square C_n)^2$$

In this section, we give the exact values of

$$\gamma_t[(P_m \square P_n)]^2, \text{ where } m, n \geq 2$$

and

$$\gamma_t[(C_m \square C_n)]^2, \text{ where } m, n \geq 3.$$

Theorem 3.4.1

For $m = 3$ or 4 or 5,

$$\gamma_t[(C_m \square C_n)]^2 = \begin{cases} 2k_2, & \text{if } n = 7k_2 - 4 \text{ or } 7k_2 - 3 \text{ or } 7k_2 - 2 \\ & \text{or } 7k_2 - 1 \text{ or } 7k_2, k_2 \geq 1, \\ 2k_2 + 1, & \text{if } n = 7k_2 + 1 \text{ or } 7k_2 + 2, k_2 \geq 1, \end{cases}$$

where

$$k_2 = \left\lceil \frac{n}{7} \right\rceil.$$

Proof :

Suppose that C_m has vertices $u_1, u_2, \ldots, u_m$ and C_n has vertices $v_1, v_2, \ldots, v_n$. Denote

$$S = \{(u_i, v_j) \in V(C_m \square C_n) \text{ such that } i = 1,2, \ldots, m \text{ and } j = 1,2,3, \ldots, n\}$$

as vertex set of $(C_m \square C_n)$ and (u_i, v_j) and (u_r, v_r) are adjacent if $u_i = u_r$ or $v_j = v_r$ and if $u_i = u_r$ and

$$v_j v_r \in E(C_n)$$

or $v_j = v_r$ and

$$u_i u_r \in E(C_m).$$

According to the values of m and n, we consider the three subcases as follows:

Subcase 1:
Let

$$n = 7k_2 - 4 \text{ or } 7k_2 - 3 \text{ or } 7k_2 - 2, k_2 \geq 1.$$

In $[(C_m \square C_n)]^2$, the subset

$$A = \{(u_3, v_{7t-6});\ 1 \leq t \leq k_2\} \cup \{(u_3, v_{7t-4});\ 1 \leq t \leq k_2\}$$

is the total dominating set. It is clear that there is no total dominating set with cardinality less than

$$|A|$$

in the graph G. Thus, A is the smallest total dominating set of $[(C_m \square C_n)]^2$. Therefore,

$$\gamma_t[(C_m \square C_n)]^2 = |A| = 2k_2.$$

Subcase 2:
Let

$$n = 7k_2 - 1 \text{ or } 7k_2, k_2 \geq 1.$$

In $[(C_m \square C_n)]^2$, the subset

$$B = \{(u_3, v_{7t-4});\ 1 \leq t \leq k_2\} \cup \{(u_3, v_{7t-2});\ 1 \leq t \leq k_2\}$$

is the total dominating set. It is clear that there is no total dominating set with cardinality less than

$$|B|$$

in the graph G. Thus, B is the minimum total dominating set of $[(C_m \square C_n)]^2$. Therefore,

$$\gamma_t[(C_m \square C_n)]^2 = |B| = 2k_2.$$

For Subcase 1 and Subcase 2, it is clear that

$$\gamma_t[(C_m \square C_n)]^2 = 2k_2$$

where

$$k_2 = \left\lceil \frac{n}{7} \right\rceil.$$

Subcase 3:
Let

$$n = 7k_2 + 1 \text{ or } 7k_2 + 2, k_2 \geq 1.$$

In $[(C_m \square C_n)]^2$, the subset

$$C = \{(u_3, v_3)\} \cup \{(u_2, v_{7t-2});\ 1 \leq t \leq k_2\} \cup \{(u_3, v_{7t});\ 1 \leq t \leq k_2\}$$

is the total dominating set. Definitely, there is no total dominating set with cardinality less than

$$|C|$$

in the graph. Thus, C is the smallest total dominating set of $[(C_m \square C_n)]^2$. Hence,

$$\gamma_t[(C_m \square C_n)]^2 = |C| = 2k_2 + 1.$$

For Subcase 3, it is clear that

$$\gamma_t[(C_m \square C_n)]^2 = 2k_2 + 1$$

where

$$k_2 = \left\lceil \frac{n}{7} \right\rceil.$$

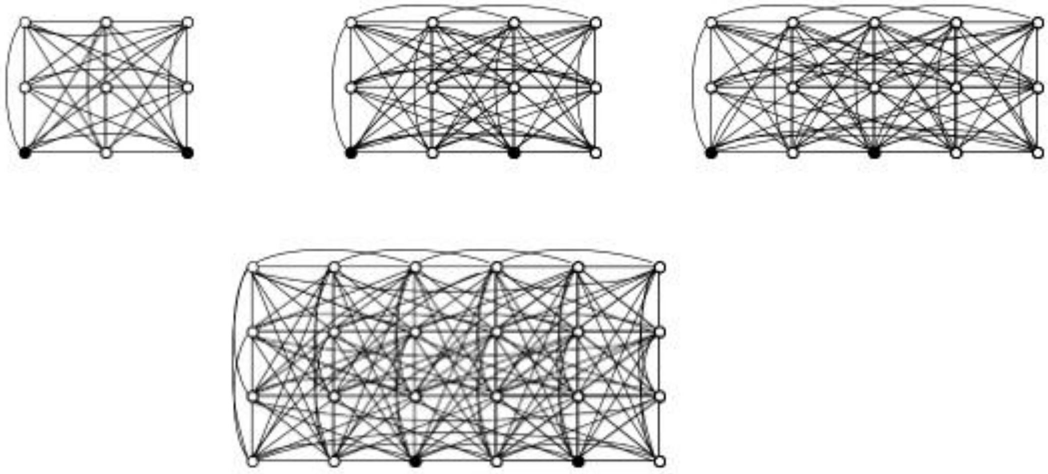

Figure 3.11: A total dominating set for $(C_3 \boxtimes C_3)^2$, $(C_3 \boxtimes C_4)^2$, $(C_3 \boxtimes C_5)^2$ and $(C_4 \boxtimes C_6)^2$

Theorem 3.4.2 (Corrected formulas only)
For $m = 6$or 7,

$$\gamma_t[(C_m \boxtimes C_n)^2] = \begin{cases} 2, & \text{if } n = 3 \text{ or } 4 \text{ or } 5, \\ 2k_2, & \text{if } n = 5k_2+1 \text{ or } 5k_2+2 \text{ or } 5k_2+3 \text{ or } 5k_2+4 \text{ or } 5k_2+5, \\ & k_2 \geq 1, \text{ where } k_2 = \left\lfloor \frac{n-1}{5} \right\rfloor. \end{cases}$$

Subcase-wise corrected formula results

- **Subcase 1**

$$\gamma_t[(C_m \boxtimes C_n)^2] = |A| = 2$$

- **Subcase 2**

$$\gamma_t[(C_m \boxtimes C_n)^2] = |B| = 2k_2$$

- **Subcase 3**

$$\gamma_t[(C_m \boxtimes C_n)^2] = |C| = 2k_2$$

$$\gamma_t[(C_m \boxtimes C_n)^2] = \begin{cases} 2, & n = 3,4,5, \\ 2k_2, & n = 5k_2 + r, r = 1,2,3,4,5, k_2 \geq 1, \\ & \text{where } k_2 = \left\lfloor \frac{n-1}{5} \right\rfloor. \end{cases}$$

Figure 3.12: A total dominating set for $(C_7 \boxtimes C_4)^2$ and $(C_7 \boxtimes C_5)^2$

Theorem 3.4.3
For $m = 8$or 9,

$$\gamma_t[(C_m \boxtimes C_n)^2] = \begin{cases} 4, & \text{if } n = 6 \text{ or } 7, \\ 3(k_2+1), & \text{if } n = 5k_2+1 \text{ or } 5k_2+2 \text{ or } 5k_2+3 \text{ or } 5k_2+4 \text{ or } 5k_2+5, \\ & k_2 \geq 2, \text{ where } k_2 = \left\lfloor \frac{n-1}{5} \right\rfloor. \end{cases}$$

Case 1

$$\gamma_t[(C_m \boxtimes C_n)^2] = |A| = 3k_2, \text{where } k_2 = \left\lfloor \frac{n}{5} \right\rfloor.$$

Case 2

$$\gamma_t[(C_m \boxtimes C_n)^2] = |B| = 4.$$

Case 3 – Subcase 1

$$\gamma_t[(C_m \boxtimes C_n)^2] = |C| = 3(k_2+1).$$

Case 3 – Subcase 2

$$\gamma_t[(C_m \boxtimes C_n)^2] = |D| = 3(k_2+1).$$

$$\gamma_t[(C_m \boxtimes C_n)^2] = \begin{cases} 4, & n = 6,7, \\ 3(k_2+1), & n = 5k_2 + r, r = 1,2,3,4,5, k_2 \geq 2, \\ & \text{where } k_2 = \left\lfloor \frac{n-1}{5} \right\rfloor. \end{cases}$$

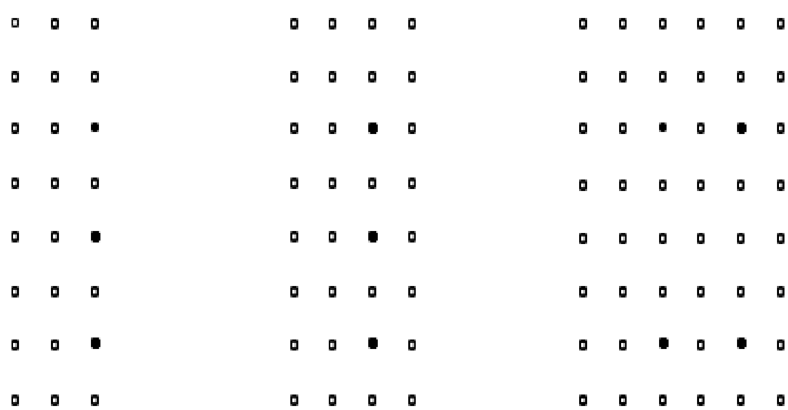

Figure 3.13: A total dominating set for $(C_8 \boxtimes C_3)^2$, $(C_8 \boxtimes C_4)^2$ and $(C_8 \boxtimes C_6)^2$

Theorem 3.4.4
For $m = 5k_1 + 1$or $5k_1 + 2$or $5k_1 + 3$or $5k_1 + 4$or $5k_1 + 5$, $k_1 \geq 2$,
$$\gamma_t[(C_m \square C_n)^2] = (k_1 + 1)(k_2 + 1) \text{if } n = 2k_2 + 4 \text{ or } 2k_2 + 5,\ k_2 \geq 1.$$

where
$$k_1 = \left\lfloor \frac{m-1}{5} \right\rfloor \text{and} k_2 = \left\lfloor \frac{n-4}{2} \right\rfloor.$$

Proof:
Suppose that $C_m$has vertices $u_1, u_2, \ldots, u_m$and $C_n$has vertices $v_1, v_2, \ldots, v_n$. Denote
$$S_i = \{(u_i, v_j) \in V(C_m \square C_n) \text{ such that } i = 1,2, \ldots, m \text{ and } j = 1,2, \ldots, n\}$$

as the vertex set of $(C_m \square C_n)$. Two vertices $(u_{i'} v_j)$and $(u_{r'} v_s)$are adjacent if $i = r$or $j = s$, and if $i = r$and $v_j v_s \in E(C_n)$, or $j = s$and $u_i u_r \in E(C_m)$.
According to the values of $m$and n, the following four subcases arise.

Subcase 1
Let $m = 5k_1 + 1$or $5k_1 + 2$, $k_1 \geq 2$, and
$n = 2k_2 + 4$, $k_2 \geq 1$.
In $(C_m \square C_n)^2$, the subset
$$A = \{(u_3, v_{2t}); 1 \leq t \leq k_2 + 1\} \cup \{(u_8, v_{2t}); 1 \leq t \leq k_2 + 1\} \cup \{(u_{11}, v_{2t}); 1 \leq t \leq k_2 + 1\}$$

is the total dominating set. It is clear that there is no total dominating set with cardinality less than $|A|$in the graph. Thus, $A$is the minimum total dominating set of $(C_m \square C_n)^2$. Hence,
$$\gamma_t[(C_m \square C_n)^2] = |A| = (k_1 + 1)(k_2 + 1).$$

Subcase 2
Let $m = 5k_1 + 3$or $5k_1 + 4$or $5k_1 + 5$, $k_1 \geq 2$, and
$n = 2k_2 + 4$, $k_2 \geq 1$.
In $(C_m \square C_n)^2$, the subset
$$B = \{(u_3, v_{2t}); 1 \leq t \leq k_2 + 1\} \cup \{(u_8, v_{2t}); 1 \leq t \leq k_2 + 1\} \cup \{(u_{11}, v_{2t}); 1 \leq t \leq k_2 + 1\}$$

is the total dominating set. Clearly, there is no total dominating set with cardinality less than $|B|$in the graph. Thus, $B$is the minimum total dominating set of $(C_m \square C_n)^2$. Hence,
$$\gamma_t[(C_m \square C_n)^2] = |B| = (k_1 + 1)(k_2 + 1).$$

Subcase 3
Let $m = 5k_1 + 1$or $5k_1 + 2$, $k_1 \geq 2$, and
$n = 2k_2 + 5$, $k_2 \geq 1$.
In $(C_m \square C_n)^2$, the subset
$$C = \{(u_3, v_{2t+1}); 1 \leq t \leq k_2 + 1\} \cup \{(u_8, v_{2t+1}); 1 \leq t \leq k_2 + 1\} \cup \{(u_{11}, v_{2t+1}); 1 \leq t \leq k_2 + 1\}$$

is the total dominating set. Clearly, there is no total dominating set with cardinality less than $|C|$in the graph. Thus, $C$is the minimum total dominating set of $(C_m \square C_n)^2$. Therefore,
$$\gamma_t[(C_m \square C_n)^2] = |C| = (k_1 + 1)(k_2 + 1).$$

Subcase 4
Let $m = 5k_1 + 3$or $5k_1 + 4$or $5k_1 + 5$, $k_1 \geq 2$, and
$n = 2k_2 + 5$, $k_2 \geq 1$.
In $(C_m \square C_n)^2$, the subset
$$D = \{(u_3, v_{2t+1}); 1 \leq t \leq k_2 + 1\} \cup \{(u_8, v_{2t+1}); 1 \leq t \leq k_2 + 1\} \cup \{(u_{11}, v_{2t+1}); 1 \leq t \leq k_2 + 1\}$$

is the total dominating set. It is clear that there is no total dominating set with cardinality less than $\mid D \mid$in the graph. Thus, $D$is the minimum total dominating set of $(C_m \square C_n)^2$. Therefore,

$$\gamma_t[(C_m \square C_n)^2] = \mid D \mid = (k_1 + 1)(k_2 + 1).$$

In all these subcases, clearly,

$$\gamma_t[(C_m \square C_n)^2] = (k_1 + 1)(k_2 + 1),$$

where

$$k_1 = \left\lfloor \frac{m-1}{5} \right\rfloor \text{and} k_2 = \left\lfloor \frac{n-4}{2} \right\rfloor.$$

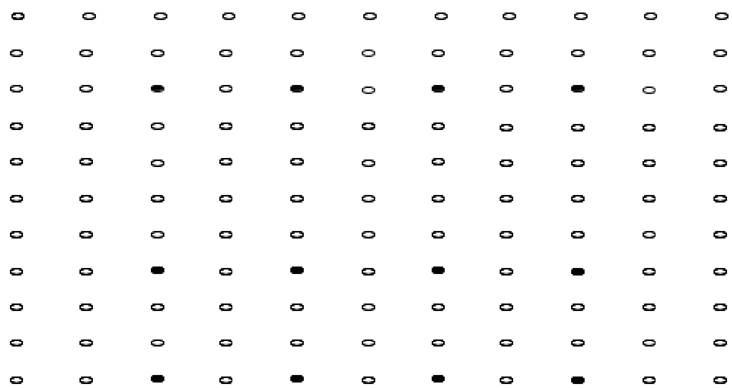

Figure 3.14: A total dominating set for $(C_{11} \boxtimes C_{11})^2$

Theorem 3.4.5
For $n = 3$or 4or 5,

$$\gamma_t[(C_m \square C_n)^2] = \begin{cases} 2(k_1 + 1), & \text{if } m = 7k_1 + 3 \text{ or } 7k_1 + 4,\ k_1 \geq 1, \\ 2k_1 + 3, & \text{if } m = 7k_1 + 5 \text{ or } 7k_1 + 6 \text{ or } 7k_1 + 7,\ k_1 \geq 1, \\ 2k_1 + 3, & \text{if } m = 7k_1 + 8 \text{ or } 7k_1 + 9,\ k_1 \geq 1, \end{cases}$$

where

$$k_1 = \left\lfloor \frac{m-1}{7} \right\rfloor.$$

Proof:
Suppose that $C_m$has vertices $u_1, u_2, \ldots, u_m$and $C_n$has vertices
$v_1, v_2, \ldots, v_n$. Denote

$$S_i = \{(u_i, v_j) \in V(C_m \square C_n) \text{ such that } i = 1,2, \ldots, m \text{ and } j = 1,2, \ldots, n\}$$

as the vertex set of $(C_m \square C_n)$. There is an edge between
$(u_i, v_j)$and (u_r, v_s)if $u_i = u_r$or $v_j = v_s$, and if
$u_i = u_r$and $v_j v_s \in E(C_n)$, or $v_j = v_s$and
$u_i u_r \in E(C_m)$.
According to the values of $m$and n, the following two cases arise.

Case 1
Let $n = 3$or 4or 5.
In this case, we consider the two subcases as follows.
Subcase 1
Let $m = 7k_1 + 3$or $7k_1 + 4$, $k_1 \geq 1$.
In $(C_m \square C_n)^2$, the subset

$$A = \{(u_3, v_3), (u_5, v_3)\} \cup \{(u_{7t+1}, v_3); 1 \le t \le k_1\} \cup \{(u_{7t+3}, v_3); 1 \le t \le k_1\}$$

is the total dominating set. Definitely, there is no total dominating set with cardinality less than $|A|$ in the graph. Thus, A is the minimum total dominating set of $(C_m \square C_n)^2$. Therefore,

$$\gamma_t[(C_m \square C_n)^2] = |A| = 2(k_1 + 1).$$

Subcase 2
Let $m = 7k_1 + 5$ or $7k_1 + 6$ or $7k_1 + 7$, $k_1 \ge 1$.
In $(C_m \square C_n)^2$, the subset

$$B = \{(u_3, v_3), (u_5, v_3)\} \cup \{(u_{7t+3}, v_3); 1 \le t \le k_1\} \cup \{(u_{7t+5}, v_3); 1 \le t \le k_1\}$$

is the total dominating set. Clearly, there is no total dominating set having cardinality less than $|B|$ in the graph. Thus, B is the minimum total dominating set of $(C_m \square C_n)^2$. Therefore,

$$\gamma_t[(C_m \square C_n)^2] = |B| = 2k_1 + 3.$$

In **Case 1**, it is clear that

$$\gamma_t[(C_m \square C_n)^2] = 2k_1 + 3, \text{where } k_1 = \left\lfloor \frac{m-1}{7} \right\rfloor.$$

Case 2
Let $n = 3$ or 4 or 5, and
$m = 7k_1 + 8$ or $7k_1 + 9$, $k_1 \ge 1$.
In $(C_m \square C_n)^2$, the subset

$$C = \{(u_3, v_3), (u_5, v_3), (u_7, v_3)\} \cup \{(u_{7t+5}, v_3); 1 \le t \le k_1\} \cup \{(u_{7t+7}, v_3); 1 \le t \le k_1\}$$

is the total dominating set. Clearly, there is no total dominating set with cardinality less than $|C|$ in the graph. Thus, C is the minimum total dominating set of $(C_m \square C_n)^2$. Hence,

$$\gamma_t[(C_m \square C_n)^2] = |C| = 2k_1 + 3.$$

In **Case 2**, it is clear that

$$\gamma_t[(C_m \square C_n)^2] = 2k_1 + 3, \text{where } k_1 = \left\lfloor \frac{m-1}{7} \right\rfloor.$$

Figure 3.15: A total dominating set for $(C_{12} \boxtimes C_4)^2$ and $(C_{12} \boxtimes C_5)^2$

Observation 3.4.1
Let P_m and P_n be path graphs with $m, n \ge 2$, and let
C_m and C_n be cycle graphs with $m, n \ge 3$. Then,

$$\gamma_t[(P_m \square P_n)^2] = \gamma_t[(C_m \square C_n)^2].$$

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